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Model Predictive Control of Linear Stochastic Systems with Constraints

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***Abstract*— In this paper, a novel model predictive control (MPC) scheme is presented for linear stochastic systems with probabilistic constraints. Instead of the prediction of the behavior of the original linear stochastic system, the behavior of a corresponding nominal linear system is predicted. Thus, the optimization problem that is solved online has the same computational burden as the ones of standard deterministic MPC of nominal systems. The control signal is specified in terms of both a nominal control action and an ancillary control law, where the ancillary control law is an optimal control law of a linear optimal stochastic control problem. Convergence of the systems in probability is discussed. The approach is illustrated with a numerical example.**

I. INTRODUCTION

The idea behind model predictive control (MPC) is to solve a finite horizon open-loop optimal control problem at each instant of time by taking the current state of the system as an initial state. The control inputs solved online are implemented in accordance with a receding horizon scheme.

Predictive control schemes have been proposed to guarantee stability and constraint fulfillment with respect to disturbances or perturbations. The aim is to determine solutions against all possible uncertainty realizations, i.e. min-max policy, where a plant family is characterized by an unknown but bounded model [1–4]. Min-max policies are often computationally demanding, and the resulting control law might be very conservative. Constraint tightening approaches [5–9] try to avoid computational complexity with a nominal prediction model and tightened constraint sets. However, the constraint sets often shrink drastically since generally the effects of uncertainties increase exponentially with the increase of the prediction horizon.

Stochastic MPC is addressed using available information on the distribution of uncertainty, where convergence in probability and expected constraints or probabilistic constraints are considered. Stochastic MPC considered multiplicative uncertainty is proposed in [10], which provides methods to handle probabilistic constraints and ensure stability through the concept of invariance in probability. Both multiplicative and additive uncertainties are considered in [11, 12], where the proposed schemes are shown to have the properties of closed-loop stability and feasibility. A key assumption in these methods is that initial states belong to an invariant set in probability by a linear or an affine state feedback law satisfying given constraints inside the set. Stochastic MPC of linear systems with process and measurement noise and bounded input policies is considered in [13], where the autonomous system is supposed to be Lyapunov stable. The optimization problem solved online is recursively feasible and the state of the system is rendered mean-square bounded.

The main contribution of this paper can be summarized as follows: the minimal variance control strategy for linear continuous-time systems with respect to white noise is investigated, which is formulated as a convex optimization problem. Boundedness of system states in probability is estimated by Multivariate Chebyshev's Inequality. MPC of linear stochastic systems is proposed which has the same computational burden as the standard MPC of deterministic systems. The recursive application of the resulting control policies renders the state of the overall system to converge to a random variable with zero mean and bounded covariance.

The paper is structured as follows. In Section II the problem setup and preliminary results are stated. The ancillary control law, the online optimization problem, the probabilistic recursive of the proposed MPC scheme, and the convergence of systems under control are discussed in Section III. A numerical example is given in Section V. Section VI concludes the paper.

A. *Notations and Basic Definitions*

For a vector $\|\cdot\|$ denote the Euclidean norm. Denote by $\|s\|_M := \sqrt{s^T M s}$ for $M = M^T \geq 0$. For any random vector s let $E[s]$ and $C_{ov}[s]$ denote the expectation and covariance matrix of s, respectively. For a symmetric matrix $X \in \mathbb{R}^{n\times n}$, let $X \succ 0 (X \succeq 0)$ denote that X is a positive (semi-) definite matrix, and $X \prec 0 (X \preceq 0)$ denote that X is a negative (semi-) definite matrix. A continuous function $\theta : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is increasing and $\theta(0) = 0$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if for each fixed s, the mapping $\beta(r, s)$ belongs to $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

II. PROBLEM SETUP AND PRELIMINARY

Consider the linear time-invariant system with state $x(t) \in \mathbb{R}^{n_x}$, control input $u(t) \in \mathbb{R}^{n_u}$, and disturbance input $w(t) \in \mathbb{R}^{n_w}$

$$\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t). \tag{1}$$

The disturbance $w(t)$, $t \geq 0$, is zero-mean, independent, identically distributed white noise, and

$$Cov\left[w_i(s), w_i(t)\right] \leq \delta(t - s)\alpha^2$$

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where $\delta(\cdot)$ is a Dirac delta function, and α is a given positive constant. The system is subject to chance constraints

$$\begin{aligned} P_r\{x(t) \in \mathcal{X}\} \geq p, \quad \forall t \geq 0 \\ P_r\{u(t) \in \mathcal{U}\} \geq p, \quad \forall t \geq 0, \end{aligned} \tag{2}$$

with $p \in (0, 1)$, where $\mathcal{X}$ and $\mathcal{U}$ denote the state constraints and input constraints, respectively.

Note that general linear constraints on states and inputs can be tackled using this framework. For instance, hard constraints can be included as a special case of chance constraints invoked with probability 1.

Assumption 1: $\mathcal{U} \subset \mathbb{R}^{n_u}$ is compact, $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ is compact and connected.

Assumption 2: The system state $x(t)$ can be measured in real time.

Assumption 3: The pair (A, B) is stabilizable, i.e., there exists a linear control law Kx such that $A+BK$ is Hurwitz. Define a nominal system

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t), \tag{3}$$

i.e., $w(t) \equiv 0$. Denote $v(t) = x(t) - \bar{x}(t)$ as the error (deviation) between the actual system (1) and the nominal system (3). The dynamics of the *error system* is given as

$$\dot{v}(t) = Av(t) + B(u(t) - \bar{u}(t)) + B_w w(t). \tag{4}$$

In this paper a control signal is designed which consists of a nominal input $\bar{u}(t)$ and a state feedback control law $K(x(t) - \bar{x}(t))$, that is,

$$u(t) = \bar{u}(t) + K\left(x(t) - \bar{x}(t)\right).$$

Choose $u(t) - \bar{u}(t) = Kv(t)$ as a state feedback control law of the system, and denote $A_{cl} = A + BK$, then the error system under the linear control law $Kv(t)$ is

$$\dot{v}(t) = A_{cl}v(t) + B_w w(t). \tag{5}$$

The main objectives of this paper are to 1) develop an MPC scheme of stochastic linear systems with chance constraints (2), 2) discuss the properties of the proposed MPC scheme.

There are two challenges inherent to our setup. First, in the presence of unbounded (i.e., Gaussian) noise, in general it is not possible to ensure any bound on the states. The additive nature of the noise ensures that the state exits from any fixed bounded set at some time [13]. Second, the hope to achieve asymptotic stability is obviously not realistic in the presence of unbounded and stochastic noise. Thus, convergence in probability to a random variable or a set is resorted to.

Before proceeding it is necessary to present the elements of probability theory and some set operations.

Definition 1: [14, 15] (Convergence in probability) A continuous random variable $X(t)$ converges in probability to a random variable X, written $X(t) \xrightarrow{\text{P}} X$, if for every $\epsilon > 0$,

$$\lim_{t\to\infty} P_r\{\|X(t) - X\| > \epsilon\} = 0. \tag{6}$$

Lemma 1: [16](Multivariate Chebyshev's Inequality) For any random vector $X \in \mathbb{R}^n$ with covariance matrix Σ,

$$P_r\left\{(X - E[X])^T\Sigma^{-1}(X - E[X]) \geq \epsilon\right\} \leq \frac{n}{\epsilon}, \tag{7}$$

for all $\epsilon \geq 0$.

Definition 2: [17] Consider two sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$. The Pontryagin difference of A and B is defined as

$$\mathcal{A} \ominus \mathcal{B} = \left\{x \in \mathbb{R}^n |\ x + y \in \mathcal{A}, \forall y \in \mathcal{B}\right\},$$

and the Minkowski sum of A and B is defined as

$$\mathcal{A} \oplus \mathcal{B} = \left\{x + y | \exists x \in \mathcal{A}, y \in \mathcal{B}\right\}.$$

Definition 3: [18] The multiplication of a set $\mathcal{B}$ by a matrix A denotes a mapping of all its elements

$$A\mathcal{B} = \left\{c |\ \exists b \in \mathcal{B}, c = Ab\right\}.$$

III. Stochastic Model predictive control

In this section a stochastic MPC scheme of linear systems is proposed. The controller has two components: a nominal control action calculated online generates a nominal and predicted state trajectory, and the ancillary control law calculated offline forces the trajectories of the error system (4) to its origin in probability, i.e. the trajectory of system (1) is forced to the nominal and predicted trajectory.

A. Ancillary Control Law

For the deterministic systems $\dot{v}(t) = Av(t) + Bh(t) + B_w w(t)$, where the disturbance is not a stochastic signal, robust control invariant set is resorted to in the formulation of the optimization problem of MPC [7–9]. In general, the robust control invariant set Ω_d of a deterministic system is a compact set.

Let us move to the stochastic system (5). Suppose that the linear feedback control law Kv has been determined. In fact, the stochastic process $\{v(t, w), t \in [0, T]\}$ is a function of two arguments of t and w. For fixed t, $v(t; \cdot)$ is a random variable and for fixed w, $v(\cdot; w)$ is thus a function of time which is a realization of the process. For $T > 0$, denote $\mathcal{W}[0, T]$ as a realization of $w(\cdot)$ in the interval of $[0, T]$. The set

$$\mathfrak{R}_w(T) := \{v(T; w(\cdot)) \mid w(\cdot) \in \mathcal{W}[0, T], v(0) = v_0\},$$

is called a reachable set of the system (5) at the time instant T. The reachable set of a stochastic system is non-convex and non-closeness [19]. Thus, it is not possible to find an enclosing set of the reachable set $\mathfrak{R}_w(T)$.
Note that with a slight abuse of notation, the stochastic process $v(t, w)$ is denoted as $v(t)$ for convenience.

Lemma 2: Consider linear system (5) where A_{cl} is stable. Suppose that $v(0) = 0$, then

$$E\left[v(t)v(t)^T\right] \preceq \alpha^2 X_c,$$

for all $t \geq 0$, where X_c is the controllability Gramian that can be obtained from the following Lyapunov equations:

$$A_{cl}X_c + X_cA_{cl}^T + B_wB_w^T = 0.$$

Proof: Due to $v(0)=0$, the covariance matrix of $v(t)$ is

$$\begin{aligned}
&E\left[v(t)v^T(t)\right]\\
=&E\Bigg[\int_0^t e^{A_{cl}(t-\tau_1)}B_w w(\tau_1)d\tau_1\cdot\\
&\qquad\left(\int_0^t e^{A_{cl}(t-\tau_2)}B_w w(\tau_2)d\tau_2\right)^T\Bigg]\\
=&E\left[\int_0^t\int_0^t e^{A_{cl}(t-\tau_1)}B_w w(\tau_1)w(\tau_2)^T B_w^T e^{A_{cl}^T(t-\tau_2)}d\tau_1 d\tau_2\right]\\
=&\int_0^t\int_0^t e^{A_{cl}(t-\tau_1)}B_w E\left[w(\tau_1)w(\tau_2)^T\right]B_w^T e^{A_{cl}^T(t-\tau_2)}d\tau_1 d\tau_2\\
\preceq&\alpha^2\int_0^t e^{A_{cl}(t-\tau_1)}B_w B_w^T e^{A_{cl}^T(t-\tau_1)}d\tau_1\\
=&\alpha^2\int_0^t e^{A_{cl}\tau}B_w B_w^T e^{A_{cl}^T\tau}d\tau
\end{aligned}$$

Since $e^{A_{cl}\tau}B_w(e^{A_{cl}\tau}B_w)^T\succeq 0$, for all $t\geq 0$,

$$E\left[v(t)v(t)^T\right]\preceq\alpha^2\int_0^\infty e^{A_{cl}\tau}B_w B_w^T e^{A_{cl}^T\tau}d\tau.$$

The lemma follows from the fact that the controllability Gramian of (A_{cl},B_w) can be represented as

$$X_c=\int_0^\infty e^{A_{cl}\tau}B_w B_w^T e^{A_{cl}^T\tau}d\tau$$

which is a solution of $A_{cl}X_c+X_cA_{cl}^T+B_wB_w^T=0$. □

The following lemma characterizes X_c in terms of linear matrix inequalities.

Lemma 3: The following statement is equivalent for the system (5) while $v(0)=0$.

(i) there exists $X_c\succ 0$ such that

$$A_{cl}X_c+X_cA_{cl}^T+B_wB_w^T=0, X_c\preceq\gamma^2 I.$$

(ii) there exists $X_a\succ 0$ such that

$$A_{cl}X_a+X_aA_{cl}^T+B_wB_w^T\preceq 0, X_a\preceq\gamma^2 I.$$

(iii) there exists $X_b\succ 0$ and Y_c such that

$$\begin{aligned}
&\begin{bmatrix}A_{cl}^TX_b+X_bA_{cl} & X_bB_w\\ B_w^TX_b & -I\end{bmatrix}\preceq 0,\\
&\begin{bmatrix}X_b & I\\ I & \gamma^2\end{bmatrix}\succeq 0
\end{aligned}\tag{8}$$

where $\gamma>0$.

Proof: (1) Since the controllability gramian is the unique positive definite solution of the Lyapunov equation $A_{cl}X_c+X_cA_{cl}^T+B_wB_w^T=0$ this is equivalent to saying that there exists X_a such that

$$A_{cl}X_a+X_aA_{cl}^T+B_wB_w^T\preceq 0, X_a\preceq\gamma^2 I.$$

(2) With a change of variables $X_b:=X_a^{-1}$, the equation above is equivalent to the existence of X_b such that

$$A_{cl}^TX_b+X_bA_{cl}+X_bB_wB_w^TX_b\preceq 0, X_b^{-1}\preceq\gamma^2 I.$$

Using Schur Complement for the two inequalities, it is equivalent to the existence of X_b such that Equ.(8) is satisfied. □

The inequality condition (8) can be formulated as a linear matrix inequality (LMI) problem [20] which is attractive since the linear control law Kv as well as the upper bound of $E[vv^T]$ can be obtained simultaneously.

Corollary 1: For the system (4), suppose that there exist a matrix $X>0$ and matrices Y such that

$$\begin{aligned}
&\begin{bmatrix}AX+BY+(AX+BY)^T & B_w\\ B_w^T & -I\end{bmatrix}\preceq 0,\\
&\begin{bmatrix}X & X\\ X & \gamma^2 I\end{bmatrix}\succeq 0.
\end{aligned}\tag{9}$$

Then, $E[vv^T]\preceq\alpha^2\gamma^2 I$ and $K=YX^{-1}$.

Proof: By substituting $X_b=X^{-1}$ and $K=YX^{-1}$ in (8) and performing a congruence transformation with the matrix $\{X,I\}$ respectively, Equ.(9) is obtained.

In terms of Lemma 2 and Lemma 3, the conclusion can be drawn directly. □

Since $w(t)$ is zero-mean and $v(0)=0$,

$$\begin{aligned}
E[v(t)]=&E\left[\int_0^t e^{A_{cl}(t-\tau)}B_w w(\tau)d\tau\right]\\
=&\int_0^t e^{A_{cl}(t-\tau)}B_w E\left[w(\tau)\right]d\tau=0.
\end{aligned}$$

Furthermore, since $C_{ov}[v]=E[vv^T]-E[v]E[v]^T$, $E[vv^T]\preceq\alpha^2X_c$ and $X_c\preceq\gamma^2 I$,

$$C_{ov}[v]\preceq\alpha^2\gamma^2 I.$$

The smallest possible upper-bound of γ^2 of the system (4) can be computed by minimizing γ^2 over all variables γ^2, X and Y that satisfy the LMI (9).

Problem 1:

$$\begin{aligned}
&\underset{X,Y,\gamma^2}{\text{minimize}}\quad\gamma^2\\
&\text{subject to}\quad(9)
\end{aligned}\tag{10}$$

B. Boundedness of system states with probabilistic p

Denote the covariance of v as Σ, i.e., $\Sigma:=C_{ov}[v]$, and $H=\alpha^2\gamma^2 I$. For fixed $\epsilon>0$, define two sets

$$\begin{aligned}
D_\Sigma&=\left\{v\in\mathbb{R}^{n_x}\mid v^T\Sigma^{-1}v\leq\epsilon\right\},\\
D_H&=\left\{v\in\mathbb{R}^{n_x}\mid v^TH^{-1}v\leq\epsilon\right\}.
\end{aligned}$$

Due to Corollary 1, $\Sigma\preceq H$ and $D_\Sigma\subseteq D_H$.

In terms of Lemma 1, $P_r\{v\in D_\Sigma\}>1-\frac{n_x}{\epsilon}$, then

$$P_r\{v\in D_H\}>1-\frac{n_x}{\epsilon}.$$

That is,

$$\begin{aligned}
P_r\{v^TH^{-1}v\leq\epsilon\}&=P_r\{v^Tv\leq\alpha^2\gamma^2\epsilon\}\\
&>1-\frac{n_x}{\epsilon}.
\end{aligned}$$

Choose $r^2:=\alpha^2\gamma^2\epsilon$, and $p=1-\frac{n_x}{\epsilon}$, the set

$$\mathcal{Z}:=\left\{v\in\mathbb{R}^{n_x}\mid v^Tv\leq r^2\right\}\tag{11}$$

is an estimate of the boundedness of system states with probability p, and

$$P_r\left\{v^Tv\leq r^2\right\}>p.$$

Furthermore, choose the sets $\mathcal{X}_0$ and $\mathcal{U}_0$ as follows

$$\begin{aligned}\mathcal{X}_0 &= \mathcal{X} \ominus \mathcal{Z}\\ \mathcal{U}_0 &= \mathcal{U} \ominus K\mathcal{Z}.\end{aligned}$$

Then, the constraints (2) can be guaranteed with probability p if $\bar{x} \in \mathcal{X}_0$ and $\bar{u} \in \mathcal{U}_0$.

C. MPC of Stochastic Systems

Nominal dynamics (3) rather than real dynamics (1) are used to predict the system behaviors, i.e. no stochastic disturbances are present. In order to guarantee satisfaction of chance constraints (2), *tightened* and *deterministic* constraints are considered.

The *online* optimization problem is formulated as follows:

Problem 2:

$$\begin{aligned}&\underset{\bar{u}(\cdot;\bar{x}(t_k))}{\text{minimize}}\ J(\bar{x}(t_k), \bar{u}(\cdot;\bar{x}(t_k)))\\ &\text{subject to}\\ &\dot{\bar{x}} = A\bar{x} + B\bar{u},\\ &\bar{x}(\tau;\bar{x}(t_k)) \in \mathcal{X}_0, \qquad \tau \in [t_k, t_k+T_p),\\ &\bar{u}(\tau;\bar{x}(t_k)) \in \mathcal{U}_0, \qquad \tau \in [t_k, t_k+T_p),\\ &\bar{x}(t_k+T_p;\bar{x}(t_k)) \in \mathcal{X}_f,\end{aligned}$$

where $\mathcal{X}_f$ is the terminal set which will be introduced in detail later. The cost function is

$$\begin{aligned}&J(\bar{x}(t_k), \bar{u}(\cdot;\bar{x}(t_k))) := S\left(\bar{x}(t_k+T_p, \bar{x}(t_k))\right) +\\ &\int_{t_k}^{t_k+T_p} \|\bar{x}\left(\tau;\bar{x}(t_k)\right)\|_Q^2 + \|\bar{u}\left(\tau;\bar{x}(t_k)\right)\|_R^2 d\tau,\end{aligned} \tag{12}$$

where T_p is the prediction horizon, $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are positive definite weighting matrices, $S(\cdot)$ is the terminal penalty.

The symbol $\bar{u}^*(\tau, \bar{x}(t_k))$, $\tau \in [t_k, t_k+T_p]$, denotes the optimal solution to Problem 2, and $\bar{x}^*(\cdot;\bar{x}(t_k))$, $\tau \in [t_k, t_k+T_p]$, is the predicted trajectory of (3) starting from $\bar{x}(t_k)$ driven by the optimal input function $\bar{u}^*(\cdot;\bar{x}(t_k))$.

Problem 2 is solved in discrete time with a sample time of δ. The nominal control during the sample interval δ is

$$\bar{u}(\tau) = \bar{u}^*\left(\tau;\bar{x}(t_k)\right), \quad \tau \in [t_k, t_k+\delta), \tag{13}$$

which is the first segment of the solution of Problem 2. The overall applied control input for the actual system (1) during the sampling interval δ consequently is

$$u(\tau) := \bar{u}(\tau) + K\left(x(\tau) - x^*(\tau;\bar{x}(t_k))\right), \tau \in [t_k, t_k+\delta).$$

The nominal controller calculated online generates a nominal state trajectory. The ancillary control law obtained offline keeps the trajectories of the error system in the set $\mathcal{Z}$ with probability p centered along the nominal trajectory.

As the pair (A, B) is stabilizable, the Riccati inequality

$$(A+BF)^T P + P(A+BF) \leq -Q - F^T R F$$

admits a solution (P, F), where $P \in \mathbb{R}^{n_x \times n_x}$ is positive definite, and $F \in \mathbb{R}^{n_u \times n_x}$. Furthermore, there exists a constant $\upsilon \in (0, \infty)$ specifying a neighborhood of the origin

$$\mathcal{X}_f := \{\bar{x} \in \mathbb{R}^{n_x} \mid \bar{x}^T P \bar{x} \leq \upsilon\}$$

where $\mathcal{X}_f \subseteq \mathcal{X}_0$ such that [21]

(1) $F\bar{x} \in \mathcal{U}_0$, for all $\bar{x} \in \mathcal{X}_f$, i.e., the linear feedback control law respects the tightened input constraints in the set $\mathcal{X}_f$,

(2) $\mathcal{X}_f$ is invariant for the nominal system controlled by the local linear feedback control law $\bar{u} = F\bar{x}$.

Denote $S(\bar{x}) := \bar{x}^T P \bar{x}$,

$$\frac{dS(\bar{x})}{dt} \leq -\bar{x}^T (Q + F^T R F)\bar{x}$$

and

$$S(\bar{x}(t)) \leq \int_t^\infty \bar{x}(s)^T (Q + F^T R F)\bar{x}(s) ds$$

Therefore, $S(x)$ and X_f can serve as the terminal penalty function and the terminal set of Problem 2, respectively [22].

Model predictive control based on the repeated solutions of Problem 2 stabilizes the nominal system [21, 23].

Associated with Problem 2, the following algorithm is implemented in this paper.

Algorithm 1:

Step 0. At time t_0, set $\bar{x}(t_0) = x(t_0)$ where $x(t_0)$ is the current state.

Step 1. At time t_k, solve Problem 2 with the current state $(\bar{x}(t_k), x(t_k))$ to obtain the nominal control action $\bar{u}(t_k)$ and the actual control action $u(t_k) = \bar{u}(t_k) + K(x(t_k) - \bar{x}(t_k))$.

Step 2. Apply the control $u(t_k)$ to the system (1) during the sampling interval $[t_k, t_{k+1}]$, where $t_{k+1} = t_k + \delta$.

Step 3. Measure the state $x(t_{k+1})$ at the next time instant t_{k+1} of the system (1) and compute the successor state $\bar{x}(t_{k+1})$ of the nominal system (3) under the nominal control $\bar{u}(t_k)$.

Step 4. Set $(\bar{x}(t_k), x(t_k)) = (\bar{x}(t_{k+1}), x(t_{k+1}))$, $t_k = t_{k+1}$, and go to Step 1.

Note that a similar algorithm was proposed in [9, 24] where tube MPC of deterministic systems is considered.

The scheme above has the same online computational complexity as the standard MPC of deterministic systems with guaranteed nominal stability [21] since only the nominal model is used for the prediction of the system dynamics and only the nominal control action is calculated online in Problem 2. The properties of the systems under control are stated in the following theorem.

Theorem 1: Suppose that F, P, $\mathcal{Z}$ and $\mathcal{X}_f$ are given, and Problem 2 is feasible at time t_0. Then,

(i) Problem 2 is feasible with probability p for all $t > t_0$,

(ii) $E\left[\lim_{t\to\infty} x(t)\right]$=0, and $C_{ov}\left[\lim_{t\to\infty} x(t)\right] \preceq \alpha^2\gamma^2 I$,

(iii) the system state $x(t)$ converges in probability to $v(t)$.

Proof: (1) For only the "computed" state and the nominal system dynamics are used to solve Problem 2 at the next time instant, the online optimization does not depend on the stochastic disturbances at all. Thus, recursive feasibility as well as constraint satisfaction is guaranteed with probability

p, provided that Problem 2 has a feasible solution at the initial time instant [21].

(2) Because of the asymptotic stability of the nominal system [21, 23], there exists a class $\mathcal{KL}$ function $\beta(\bar{x}, t)$ [25] such that

$$\|\bar{x}(t)\| \leq \beta(\bar{x}(t_0), t), \quad \forall t \geq t_0.$$

Furthermore, since $\beta(\cdot, \cdot)$ is a $\mathcal{KL}$ function, for all $\varpi > 0$, there exists t_ϖ such that for all $t \geq t_\varpi$

$$\|\bar{x}(t) - 0\| = \|\bar{x}(t)\| \leq \beta(\bar{x}(t_0), t) \leq \varpi.$$

That is, $\lim_{t\to\infty} \bar{x}(t) = 0$.
For $x(t) = \bar{x}(t) + v(t)$, $E[v(t)] = 0$ and $C_{ov}[v(t)] \preceq \alpha^2\gamma^2 I$

$$\begin{aligned} E\left[\lim_{t\to\infty} x(t)\right] &= \lim_{t\to\infty} \bar{x}(t) + E\left[\lim_{t\to\infty} v(t)\right] \\ &= \lim_{t\to\infty} \bar{x}(t) + \lim_{t\to\infty} E\left[v(t)\right] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} C_{ov}\left[\lim_{t\to\infty} x(t)\right] &= C_{ov}\left[\lim_{t\to\infty} \bar{x}(t)\right] + C_{ov}\left[\lim_{t\to\infty} v(t)\right] \\ &= \lim_{t\to\infty} C_{ov}\left[\bar{x}(t)\right] + \lim_{t\to\infty} C_{ov}\left[v(t)\right] \\ &\preceq \alpha^2\gamma^2 I. \end{aligned}$$

(3) Since $v(t) = x(t) - \bar{x}(t)$, and $\|\bar{x}(t)\| \leq \beta(\bar{x}(t_0), t)$ for all $t \geq t_0$,

$$\lim_{t\to\infty} [x(t) - v(t)] = 0.$$

Thus, for every $\epsilon > 0$,

$$\lim_{t\to\infty} P_r\left\{\|x(t) - v(t)\| > \epsilon\right\} = 0, \tag{14}$$

i.e., $x(t)$ converges in probability to the random variable $v(t)$. □

IV. Discussion on Non-zero-mean disturbances

Suppose that $E[w(t)] = c(t) \in \mathbb{R}^{n_w}$. For $E[v]E[v]^T \succeq 0$, $C_{ov}[v] \preceq \alpha^2\gamma^2 I$, which is not necessarily a supremum of $C_{ov}[v]$. In terms of $v(t) = \int_0^t e^{A_{cl}(t-\tau)} B_w w(\tau) d\tau$,

$$\begin{aligned} E[v(t)] &= \int_0^t e^{A_{cl}(t-\tau)} B_w E[w(\tau)] d\tau \\ &= \int_0^t e^{A_{cl}(t-\tau)} B_w c(\tau) d\tau \end{aligned}$$

Since A_{cl} is Hurwitz, there are constants $M \geq 1$ and $\beta < 0$ such that [26]

$$\|e^{A_{cl}t}\| \leq M e^{\beta t}, \forall t \geq 0.$$

Denote $\sup_{\tau\in[0,\infty)} \|c(\tau)\| := c_{\max}$,

$$\begin{aligned} \|E[v(t)]\| &\leq \int_0^t M e^{\beta(t-\tau)} \|B_w\| \cdot \|c(\tau)\| d\tau \\ &\leq \int_0^t M e^{\beta(t-\tau)} \|B_w\| c_{\max} d\tau \\ &= \frac{c_{\max} M \|B_w\|}{-\beta}. \end{aligned}$$

Furthermore, denote $\bar{r} := r + \frac{c_{\max} M \|B_w\|}{-\beta}$, and define a polytopic set

$$\bar{\mathcal{Z}} := \{v \in \mathbb{R}^{n_x} \mid v_i \in [-\bar{r}, \bar{r}]\}.$$

The sets $\mathcal{X}_0$ and $\mathcal{U}_0$ can be chosen as follows

$$\begin{aligned} \mathcal{X}_0 &= \mathcal{X} \ominus \bar{\mathcal{Z}} \\ \mathcal{U}_0 &= \mathcal{U} \ominus K\bar{\mathcal{Z}}. \end{aligned}$$

Therefore, the constraints (2) can be guaranteed with probability p if $\bar{x} \in \mathcal{X}_0$ and $\bar{u} \in \mathcal{U}_0$.

V. Illustrative example

Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ -8 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t), \tag{15}$$

where x_1 and x_2 can be measured instantaneously, and the control input $u \in \mathbb{R}^1$. The disturbance $w \in \mathbb{R}^1$ is a gaussian random variable with $E[w] = 0$ and $var(w) = 0.3$.

The system is subject to the chance constraint

$$P_r\{-2 \leq u(t) \leq 2\} \geq 0.97, \forall t \geq 0. \tag{16}$$

Solving Problem 1 to get $K = \begin{bmatrix} -0.4318 & 1.0288 \end{bmatrix}$ and $\gamma^2 = 0.1987$. Thus, $C_{ov}[v(t)] \leq 0.0179I$. Furthermore, $r = 0.3821$ and $\mathcal{Z} := \{v \in \mathbb{R}^{n_x} \mid v_i \in [-0.3821, 0.3821]\}$, see Equ. (11). Here a polytopic set rather than a circle is chosen to bound the effect of white noise.

Choose the stage cost function as $l(\bar{x}, \bar{u}) = \bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u}$, where the penalty matrices $Q = \text{diag}(0.5, 0.5)$, $R = 1$. Both the terminal control law and the terminal penalty matrix are computed by the solution of a convex optimization problem, c.f. [27], $F\bar{x} = \begin{bmatrix} -0.3049 & 0.3681 \end{bmatrix} \bar{x}$ and $S(\bar{x}) = \bar{x}^T \begin{bmatrix} 1.4425 & -1.7215 \\ -1.7215 & 3.6844 \end{bmatrix} \bar{x}$. The terminal set of the optimization problem is $\mathcal{X}_f = \{\bar{x} \in \mathbb{R}^{n_2} \mid S(\bar{x}) \leq 10\}$. The open-loop optimization problem described by Problem 2 is solved in discrete time with a sampling time of $\delta = 0.1$ time units and a prediction horizon of $T_p = 1.5$ time units.

Figure 1 shows the state trajectory starting from state $[-8.0 \ \ -8.0]^T$ with the stochastic disturbances $w(t)$, where the simulation is run 20 times. As can be seen, the chance constraint (16) has never been violated although the "worst" disturbance is $|w| = 0.8552$. It is mainly because controllability gramian is used to bound the covariance of $v(t)$ for all $t \in [0, \infty)$. Furthermore, the system state converges to the set $\mathcal{Z}$ in finite time.

VI. Conclusions

MPC of continuous-time linear systems subject to white noises and chance constraints was derived in this paper. An error system which is the deviation of the actual system from the nominal system was defined. Ancillary control law determined off-line keeps the error system to lie in a set in probability. The optimization problem solved online has the same computational burden as the standard MPC of deterministic systems with guaranteed nominal stability. The solution of the optimization problem defines the nominal

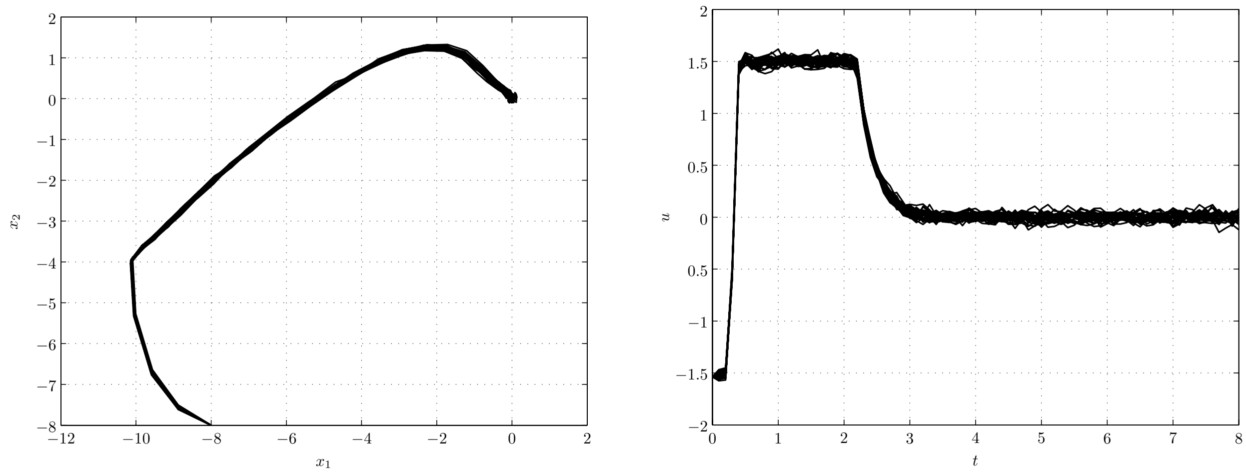

Fig. 1. Exemplary time profiles for the closed-loop system (15) with Algorithm 1 for disturbances w from $x_0 = [-8.0 \;\; -8.0]^T$, where $E[w] = 0$ and $var(w) = 0.3$.

trajectory. The actual trajectory of the system under the proposed MPC control law is in a set along the nominal trajectory in probability. Moreover, both recursive feasibility of the online optimization problem and convergence of the system in probability are guaranteed if the optimization problem is feasible at the initial time instant.

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